Stochastic Interpretation for The Dirichlet-Poisson Problem with Measurable Drift

Liangquan Zhang[1,2] *
1. School of Mathematics, Shandong University
Jinan 250100, People's Republic of China.
2. Laboratoire de Mathématiques,
Université de Bretagne Occidentale, 29285 Brest Cédex, France.

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Abstract

In this article, we show the stochastic interpretation for the solution of the Dirichlet-Poisson Problem with bounded, measurable drift and prove it is a viscosity solution as well.

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1 Introduction

At the beginning, let $\mathcal{D}$ be a domain (open connected set) in $\mathbb{R}^n$ and let $\mathcal{L}$ denote a elliptic partial differential operator of the following form

$$\mathcal{L} = \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} \tag{1.1}$$

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where $b_i(x)$ is a measurable and bounded function while $a_{i,j}(x) = a_{j,i}(x)$ is a continuous functions (see below). Now consider the following combined Dirichlet-Poisson Problem:

$$\begin{cases} \mathcal{L}w = -g, & \text{in } \mathcal{D}, \\ \lim\limits_{x\to y,\, x\in\mathcal{D}} w(x) = \Psi(y), & y \in \partial\mathcal{D}, \end{cases} \tag{1.2}$$

where $g \in C(\mathcal{D})$ and $\Psi \in C(\partial\mathcal{D})$ are given functions. When b_i and $a_{i,j}$ satisfy Lipschitz condition and other some smooth conditions, $\mathcal{L}$ is uniformly elliptic in $\mathcal{D}$, then if the function $w \in C^2(\mathcal{D})$ solves the Dirichlet-Poisson Problem, it can be expressed by

$$w(x) = \mathbb{E}^x\left[\Psi\left(X_{\tau_\mathcal{D}}\right)\right] + \mathbb{E}^x\left[\int_0^{\tau_\mathcal{D}} g(X_s)\,\mathrm{d}s\right], \qquad x \in \mathcal{D}, \tag{1.3}$$

which, of course, is also a viscosity solution. (see Theorem 9.3.3 in [4]).

A natural question arises is that when the coefficient b is only measurable and bounded, and w is a unique weak solution (for example, Sobolev or viscosity solution), do we have above expression (1.3)? Our answer is positive.

This paper is organized as follows: after giving the preliminaries in Section 2, we are devote to showing the stochastic interpretation of the solution of the Dirichlet-Poisson Problem and prove it is also a viscosity solution as well.

2 Preliminaries

Let $v = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, W\right)$ be a reference probability system composed of a completed probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions of right-continuity and completeness, and a d-dimensional $(\mathcal{F}_t)$-Brownian motion W defined on $(\Omega, \mathcal{F}, P)$.

Let

$$W^{k,p}(\mathcal{D}) = \left\{u \in W^k(\mathcal{D});\, D^\alpha u \in L^p(\mathcal{D}) \text{ for all } |\alpha| \leq k\right\}.$$

The space is obviously linear. A norm is introduced by defining

$$\|u\|_{k,p;\mathcal{D}} = \|u\|_{W^{k,p}(\mathcal{D})} = \left(\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u|^p\right)^{\frac{1}{p}}. \tag{2.0}$$

Note that $W^{k,p}(\mathcal{D})$ is a Banach space under norm (2.0). Another Banach space $W_0^{k,p}(\mathcal{D})$ arisen by taking the closure of $C_0^k(\mathcal{D})$ in $W^{k,p}(\mathcal{D})$. The space $W^{k,p}(\mathcal{D})$, $W_0^{k,p}(\mathcal{D})$ do not coincide for bounded $\mathcal{D}$. Let $C(\mathcal{D})$ denote the set of continuous functions on $\mathcal{D}$.

Consider the following SDEs:

$$\begin{cases} \mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \\ X(0) = x. \end{cases} \tag{2.1}$$

We assume the following conditions:

A1: There exist a constant $K \geq 0$, such that

$$|b(x)| + |\sigma(x)| \leq K,\ \forall x \in \mathbb{R}^n$$

for dx-almost every $x \in \mathbb{R}^n$.

A2: σ is a continuous function from $\mathbb{R}^n \to \mathbb{R}^{n\times d}$, satisfies the uniformly elliptic condition, i.e.,

$$\exists \lambda > 0; \qquad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,\ \xi^* \sigma \xi \geq \lambda |\xi|^2,$$

where ξ^* denotes the adjoint of ξ.

A3: Assume that

$$\sigma \in W^{1,2(n+1)}_{loc}(\mathcal{D}).$$

Let recall some result from [1].

Proposition 1. *Suppose that assumptions (A2)-(A3) are satisfied. Then SDEs (2.1) has one and only one strong solution.*

For simplicity, let $\mathcal{K}_R$ be a unit ball with the radius $R > 0$. Now define

$$\tau(x) \doteq \inf\{t \geq 0 : X(t) \notin \mathcal{K}_R\},\ \forall x \in \mathcal{K}_R \tag{2.2}$$

and

$$\mathcal{U}(x) = \mathbb{E}\left[\Psi(X(\tau(x))) + \int_0^{\tau(x)} g(X(s))\,\mathrm{d}s\right], \quad x \in \mathcal{K}_R. \tag{2.3}$$

Let us recall

Proposition 2 (Theorem 9.15 and Corollary 9.18 in [3])**.** *Assume that (A1)-(A3) hold. Then the Dirichlet-Poisson Problem*

$$\begin{cases} \mathcal{L}w = -g, & in\ \mathcal{K}_R, \\ w - \Psi \in W^{1,p}_0(\mathcal{K}_R), \\ w(x) = \Psi(y), & y \in \partial\mathcal{K}_R, \\ \Psi \in W^{2,p}(\mathcal{K}_R),\ g \in L^p(\mathcal{K}_R), \end{cases} \tag{2.4}$$

has a unique strong solution $u \in W^{2,p}(\mathcal{K}_R) \bigcap W^{2,p}_{loc}(\mathcal{K}_R) \bigcap C\left(\overline{\mathcal{K}_R}\right)$, $p > n$.

3 Main Result

First of all, we have the following Krylov's estimate.

Lemma 3. *Suppose that assumptions (A1)-(A3) are satisfied. Then for any Borel function* $f \in L^q(\mathcal{K}_R)$, *and* $q > n+2$, *we have*

$$\mathbb{E}\left[\int_0^{T\wedge\tau(x)} f(X(t))\,dt\right] \leq N\|f\|_{L^q(\mathcal{K}_R)}, \tag{3.1}$$

where N *is a constant depending only on* $\mathcal{K}_R$, $T>0$ *and* K.

We omit the proof of Lemma 3 since it is very similar to that of Theorem 3 in [5]. The following result generalizes Krylov's extension of Itô's formula.

Lemma 4. *Assume that (A1)-(A3) holds. Then for any* $u:\mathbf{R}^n \to \mathbf{R}$ *from the Sobolev space* $W^{2,p}(\mathcal{K}_r)$, $p>n+2$ *we have*

$$u(X(t)) - u(x) = \int_0^t (\mathcal{L}u(X(s)))\,ds + \int_0^t \partial_x u(X(s))\,\sigma(X(s))\,dW_s, \tag{3.2}$$

almost surely for $t \leq \tau(x)$.

Remark 5. *Because the diffusion term is not degenerated, the* τ *is almost surely finite.*

Proof. We first show that each integral in (3.2) is well-defined. By virtue of Sobolev's embedding theorem there exists a constant N such that

$$\sup_{x\in\mathcal{K}_R}\left(|u(x)| + \sum_i |\partial_{x_i}u(x)|\right) \leq N\|u\|_{W^{2,p}(\mathcal{K}_R)}.$$

for all $u \in W^{2,p}(\mathcal{K}_R)$, $p>n+2$. Hence, for any $T>0$,

$$\mathbb{E}\left[\int_0^{\tau(x)\wedge T} |\partial_x u(X^{x,\varepsilon}(s))|^2\,\mathrm{d}s\right] \leq KT\|u\|^2_{W^{2,p}(\mathcal{K}_R)}, \tag{3.3}$$

and

$$\mathbb{E}\left[\int_0^{\tau(x)\wedge T} |b_i(X^{x,\varepsilon}(s))\,\partial_{x_i}u(x)|\,\mathrm{d}s\right] \leq KT\|u\|_{W^{2,p}(\mathcal{K}_R)}, \tag{3.4}$$

by the boundness of b and σ. Moreover,

$$\mathbb{E}\left[\int_0^{\tau(x)\wedge T} \left|\partial_{x_i,x_j}u(X^{x,\varepsilon}(s))\right|\mathrm{d}s\right] \leq N\left\|\partial_{x_i,x_j}u\right\|_{L^p(\mathcal{K}_R)} \leq NT\|u\|_{W^{2,p}(\mathcal{K}_R)}, \tag{3.5}$$

Consequently, the right-hand side of (3.2) is well-defined for $t \leq T\wedge\tau(x)$.

As a matter of fact, for $u \in W^{2,p}(\mathcal{K}_R)$ there exists a sequence of function u_n in $C^2(\mathbb{R}^n)$ such that

$$\|u_n - u\|_{W^{2,p}(\mathcal{K}_R)} \to 0, \qquad \text{as } n\to+\infty.$$

Applying Itô's formula we have

$$\begin{aligned}
& u_n\left(X\left(t \wedge \tau\left(x\right) \wedge T\right)\right) - u_n\left(x\right) \\
= & \int_0^{t \wedge \tau(x) \wedge T} \left(\mathcal{L} u_n\left(X^{x,\varepsilon}\left(s\right)\right)\right) \mathrm{d}s + \int_0^{t \wedge \tau(x) \wedge T} \partial_x u_n\left(X\left(s\right)\right) \sigma\left(X\left(s\right)\right) \mathrm{d}W_s.
\end{aligned} \tag{3.6}$$

for any $t \geq 0$. On the other hand, the inequalities (3.3)-(3.5) hold with $u - u_n$ replacing of u, with constants independent of n. Lastly, letting n, $T \to +\infty$ in (3.6) we get the desired result. □

We are now in a position to make an important observation as follows:

Theorem 6. *Under the assumption (A1)-(A3), let* $u \in W_0^{1,p}\left(\mathcal{K}_R\right) \cap W^{2,p}\left(\mathcal{K}_R\right) \cap C\left(\mathcal{K}_R\right)$, $p > n+2$, *be a unique solution of (2.4). Then we have*

$$u\left(x\right) = \mathcal{U}\left(x\right), \qquad \forall x \in \mathcal{K}_R.$$

Proof. Applying Lemma 4 to u and taking the expectation, we have

$$\begin{aligned}
& \mathbb{E}\left[u\left(X\left(\tau\left(x\right)\right)\right) - u\left(x\right)\right] \\
= & \mathbb{E}\left[\int_0^{\tau(x)} \mathcal{L} u\left(X\left(s\right)\right) \mathrm{d}s + \int_0^{\tau(x)} g\left(X\left(s\right)\right) \mathrm{d}s\right. \\
& \left. - \int_0^{\tau(x)} g\left(X\left(s\right)\right) \mathrm{d}s + \int_0^{\tau(x)} \partial_x u\left(X\left(s\right)\right) \sigma\left(X\left(s\right)\right) \mathrm{d}W_s\right].
\end{aligned}$$

Since

$$\begin{cases} \mathbb{E}\left[\mathcal{L} u\left(X\left(s\right)\right) + g\left(X\left(s\right)\right)\right] = 0, \\ \mathbb{E}\left[\int_0^{\tau(x)} \partial_x u\left(X\left(s\right)\right) \sigma\left(X\left(s\right)\right) \mathrm{d}W_s\right] = 0, \ 0 \leq s \leq \tau\left(x\right), \end{cases}$$

we obtain

$$u\left(x\right) = \mathcal{U}\left(x\right), \qquad \forall x \in \mathcal{K}.$$

The proof is complete. □

Now let us introduce the following definitions of viscosity solution for second-order PDE from [2]:

$$F\left(x, V, DV, D^2 V\right) = 0, \quad x \in \Omega, \tag{3.7}$$

where $\Omega \subset \mathbb{R}^n$ denotes an open set.

Definition 7. *(i) We say that* $V \in C\left(\Omega\right)$ *is a viscosity subsolution of (3.7) at a point* $x_0 \in \Omega$, *if and only if, for any test function* $\varphi \in C^2\left(\Omega\right)$ *such that* $V - \varphi$ *has a local maximum at* x_0, *then*

$$F\left(x_0, V\left(x_0\right), D\varphi\left(x_0\right), D^2\varphi\left(x_0\right)\right) \leq 0; \tag{3.8}$$

(ii) We say that $V \in C(\Omega)$ is a viscosity supersolution of (3.7) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $V - \varphi$ has a local minimum at x_0, then

$$F\left(x_0, V(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \geq 0; \tag{3.9}$$

(iii) We say that V is a viscosity solution in the open set Ω if V is a viscosity subsolution and a viscosity supersolution, at any point $x_0 \in \Omega$.

We will characterize the value function (2.3) as a unique viscosity solution of the PDE (2.4).

Theorem 8. *Under assumptions (A1)-(A3). Then, u is a unique viscosity solution of the Dirichlet-Poisson equation (2.4).*

Proof. The uniqueness is obtained from Proposition 2 . We shall prove u is the viscosity solution of (2.4). Obviously, $u(x) = \Psi(x)$, $x \in \partial\mathcal{K}$. Let us show in a first step that u is a viscosity super-solution. For this we suppose that, for all compactly-supported $C^2(\mathbb{R}^n;\mathbb{R})$ (twice differentiable with continuous second derivatives) function $\varphi : \mathbb{R}^n \to \mathbb{R}$, whenever x_0 is a point of local minimum of $u - \varphi$, that is

$$\begin{cases} \varphi(x_0) = u(x_0), \\ \varphi(y) \leq u(y), \quad y \neq x_0. \end{cases}$$

Then, thanks to the pathwise uniqueness of SDEs (2.1), we have

$$\mathbb{E}\left[\varphi(X(t))\right] - \varphi(x_0) \leq -\mathbb{E}\left[\int_0^t g(X(s))\,\mathrm{d}s\right].$$

By virtue of Dynkin's formula, we have

$$\sum_{i=1}^n b(x_0)\frac{\partial}{\partial x_i}\varphi(x_0) + \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x_0) + g(x_0) \leq 0.$$

Similarly, the sub-solution is proved. The proof is complete. □

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